Jordan's theorem on primitive groups involving a p-cycle

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Question

If G is a group with a 2-transitive action on Ω, then G is primitive on Ω.

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No.

Example

Consider D_5. Let Ω be the set of cosets of the subgroup H of order 2. Observe that D_5 acts transitively on Ω by right multiplication on Ω. Since $|\Omega| = 5$, this group acts primitively on Ω. But this group action cannot be 2-transitive, otherwise, D_5 would be transitive on the set of ordered par of distinct points on Ω. Since $20 \nmid |D_5|$, and this is a contradiction.

Question

Question: if G is primitive, then G is a group with a 2-transitive action on Ω?

No.

However, one condition is enough for a positive response.

Question

Theorem

Let G be a group that acts primitively on a finite set Ω. Let $\Lambda \subseteq \Omega$ be such that $|\Lambda| \leq |\Omega| - 2$. If G_Λ acts primitively on $\Omega - \Lambda$, then the action of G on Ω is $(|\Lambda| + 1)$-transitive.

Question

Theorem

Let G be a group that acts primitively on a finite set Ω. Let $\Lambda \subseteq \Omega$ be such that $|\Lambda| \leq |\Omega| - 2$. If G_Λ acts primitively on $\Omega - \Lambda$, then the action of G on Ω is $(|\Lambda| + 1)$-transitive.

- This result is due to Camille Jordan in 1870's.
- It will be important to demonstrate the main theorem of this work.

Main theorem

Theorem

Let G be a primitive permutation group on a finite set Ω. Let p be a prime with $p \leq |\Omega| - 3$. If G contains a p-cycle, then G is either the alternating group or the symmetric group.

Main theorem

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Let G be a primitive permutation group on a finite set Ω. Let p be a prime with $p \leq |\Omega| - 3$. If G contains a p-cycle, then G is either the alternating group or the symmetric group.

- This result was also assigned to Camille Jordan.
- It is a result of classification of primitive permutation groups on finite sets that contains a p-cycle.

Translate and block

Definition

- Let G be a group that acts transitively on a set Ω.
- $\varnothing \neq \Delta \subseteq \Omega$.

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The set $\Delta g = \{\delta g \mid \delta \in \Delta\}$ is called a **translate** of Δ, with $g \in G$.

Translate and block

Definition

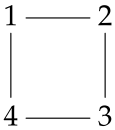
- Let G be a group that acts transitively on a set Ω.
- $\varnothing \neq \Delta \subseteq \Omega$.

If $\forall g \in G$ such that $\Delta g \neq \Delta$ we have $\Delta g \cap \Delta = \varnothing$, we say that Δ a **block**.

Translate and block

Example

Let D_4 acting on $\Omega = \{1, 2, 3, 4\}$ and let $\Delta = \{1, 3\}$. Then $\Delta g = \{1, 3\}$ or $\{2, 4\}$, that is, Δ is a block.



Primitive and imprimitive

Definition

A transitive action of G on Ω is **primitive** if the only blocks are

$$\Omega \quad \text{and} \quad \{\omega\}, \text{ for all } \omega \in \Omega.$$

Otherwise, the action is **imprimitive.**

Properties of block

Lemma

Let G be a group that acts transitively on a finite set Ω and let $\Delta \subseteq \Omega$ be a block. Denote $\mathcal{P} = \{\Delta g \mid g \in G\}$. Then,

- *Δg is a block, that is, for all $g, h \in G$ such that $\Delta g \neq \Delta h$, we have $\Delta g \cap \Delta h = \varnothing$.*

Properties of block

Lemma

Let G be a group that acts transitively on a finite set Ω and let $\Delta \subseteq \Omega$ be a block. Denote $\mathcal{P} = \{\Delta g \mid g \in G\}$. Then,

- *G acts transitively on $\mathcal{P}$.*

Properties of block

Lemma

Let G be a group that acts transitively on a finite set Ω and let $\Delta \subseteq \Omega$ be a block. Denote $\mathcal{P} = \{\Delta g \mid g \in G\}$. Then,

- $\cup_{g \in G} \Delta g = \Omega$.

Properties of block

Lemma

Let G be a group that acts transitively on a finite set Ω and let $\Delta \subseteq \Omega$ be a block. Denote $\mathcal{P} = \{\Delta g \mid g \in G\}$. Then,

- $|\Delta|$ *divides* $|\Omega|$ *and* $|\mathcal{P}| = \frac{|\Omega|}{|\Delta|}$.

Properties of block

Corollary

A transitive group action on a set of prime cardinality is primitive.

Action k-transitive

Definition

Let G be a group that acts on a set Ω where $|\Omega| \geq k$. We say that G is **k-transitive on Ω**, if

$$\forall\,(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_k),(\beta_1,\ \beta_2,\ \ldots,\ \beta_k) \in \Omega^k$$

with $\alpha_i \neq \alpha_j$ and $\beta_i \neq \beta_j$, if $i \neq j$, there exists an element $g \in G$ such that

$$\alpha_i g = \beta_i, \text{ for } 1 \leq i \leq k$$

If $k > 1$, then we say that an action is **multiply transitive.**

Action k-transitive

Observation

If G is k-transitive on Ω, then G is ℓ-transitive on Ω, for all $1 \leq \ell \leq k$.

Action k-transitive

Example
The group S_n is n-transitive on $\Omega = \{1, 2, \dots, n\}$.

Furthermore, the only subgroups of S_n that are $(n-2)$-transitive on $\{1, ..., n\}$ are S_n and A_n.

Pointwise and setwise stabilizer

Definition

The **pointwise stabilizer** of Λ is defined as

$$G_\Lambda := \{g \in G \mid \lambda g = \lambda,\ \forall\, \lambda \in \Lambda\}.$$

The **setwise stabilizer** of Λ is defined as

$$G_{(\Lambda)} := \{g \in G \mid \Lambda g = \Lambda\}.$$

Pointwise and setwise stabilizer

Observation

1. $G_{(\Lambda)}$ acts on Λ and the kernel of this action is G_{Λ}. Thus $G_{\Lambda} \trianglelefteq G_{(\Lambda)}$.
2. Furthermore $G_{(\Lambda)} = G_{(\Omega-\Lambda)}$ that implies $G_{\Omega-\Lambda} \trianglelefteq G_{(\Lambda)}$.

Lemma

Suppose that a group G acts transitively on a finite set Ω. A group G is k-transitive on Ω if and only if the stabilizer G_γ is $(k-1)$-transitive on $\Omega - \{\gamma\}$, where $\gamma \in \Omega$ and k is an integer such that $k \leq |\Omega|$.

Definition

Given a transitive action of G on Ω and a set $X \subsetneq \Omega$. We say that X is a **Jordan set** if G_X is transitive on $\Omega - X$, and that X is **strongly Jordan set** if G_X is primitive on $\Omega - X$.

Theorem

Let G be a group acting primitively on a finite set Ω and let $X \subseteq \Omega$ be a Jordan set with $0 < |X| < |\Omega| - 1$. Then, for all $\alpha \in \Omega$, the stabilizer G_α is transitive on $\Omega - \{\alpha\}$ and G is 2-transitive on Ω. Furthermore, if X is strongly Jordan, then G_α is primitive on $\Omega - \{\alpha\}$.

Proof Sketch

It is sufficient to prove that under this conditions every one-point subset of Ω is (strongly) Jordan set.

Let X_0 be the minimal among nonempty (strongly) Jordan subsets of X.

For the cases $|X_0| \geq \frac{|\Omega|}{2}$ and $1 < |X_0| < \frac{|\Omega|}{2}$, we obtain a contradiction.

Since translates of (strongly) Jordan set are (strongly) Jordan set, thus we obtain the result.

Jordan's theorem

Theorem

Let G be a group that acts primitively on a finite set Ω. Let $\Lambda \subseteq \Omega$ be such that $|\Lambda| \leq |\Omega| - 2$. Suppose that G_Λ acts primitively on $\Omega - \Lambda$. Then the action of G on Ω is $(|\Lambda| + 1)$-transitive.

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Let G be a group that acts primitively on a finite set Ω. Let $\Lambda \subseteq \Omega$ be such that $|\Lambda| \leq |\Omega| - 2$. Suppose that G_Λ acts primitively on $\Omega - \Lambda$. Then the action of G on Ω is $(|\Lambda| + 1)$-transitive.

Proof Sketch

We will prove by induction on $|\Omega|$. We can assume $|\Omega| > 2$ and $|\Lambda| > 0$.

For $|\Omega| = 3$ we have $|\Lambda| = 1$ and the result is true. Suppose the result is valid for $|\Omega| < k$. Let $|\Omega| = k > 3$ and let $\alpha \in \Omega$.

We have G_α is primitive on $\Omega - \{\alpha\}$. Then we can assume $\alpha \in \Lambda$.

Jordan's theorem

Proof Sketch

$$\begin{aligned}(G_\alpha)_{\Lambda-\{\alpha\}} &= \{g \in G_\alpha \mid \lambda \cdot g = \lambda,\ \forall \lambda \in \Lambda - \{\alpha\}\}.\\ &= \{g \in G \mid \alpha \cdot g = \alpha \wedge \lambda \cdot g = \lambda,\ \forall \lambda \in \Lambda - \{\alpha\}\}\\ &= G_\Lambda.\end{aligned}$$

Jordan's theorem

Proof Sketch

By the hypothesis G_Λ is primitive on $\Omega - \Lambda = (\Omega - \{\alpha\}) - (\Lambda - \{\alpha\})$.
We have $|\Lambda - \{\alpha\}| \leq |\Omega - \{\alpha\}| - 2$, since $|\Omega - \{\alpha\}| = k - 1$

We can apply the inductive hypothesis. Then the action of G_α to $\Omega - \{\alpha\}$ is $|\Lambda|$-transitive. Thus, we obtain that G is $(|\Lambda| + 1)$-transitive.

Jordan's theorem involving a 3-cycle

Corollary

Let G be a primitive permutation group on a finite set Ω that contains a 3-cycle. Then G is either the symmetric group or the alternating group on Ω.

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Let G be a primitive permutation group on a finite set Ω that contains a 3-cycle. Then G is either the symmetric group or the alternating group on Ω.

Proof Sketch

Let (a, b, c) be a 3-cycle contained in G. Let $\Lambda = \Omega - \{a, b, c\}$.

From the definition of G_Λ we have $(a, b, c) \in G_\Lambda$, that implies G_Λ is transitive on $\{a, b, c\} = \Omega - \Lambda$.

Jordan's theorem involving a 3-cycle

Since $|\{a, b, c\}| = 3$ is a prime number, we obtain that G_Λ is primitive on $\{a, b, c\}$. We have $|\Lambda| \leq |\Lambda| + 1 = |\Omega| - 2$, then we have G is $(|\Omega| - 2)$-transitive on Ω.

The only subgroups of $\mathrm{Sym}(\Omega)$ that are $(n-2)$-transitive on Ω are $\mathrm{Sym}(\Omega)$ and $\mathrm{Alt}(\Omega)$

Lemma

Let x be an n-cycle in S_n. Then $\langle x \rangle = C_{S_n}(x) = \{g \in S_n \mid gx = xg\}$.

Jordan's theorem involving a p-cycle

Theorem (Jordan)

Let G be a primitive permutation group on a finite set Ω. Let p be a prime with $p \leq |\Omega| - 3$. If G contains a p-cycle, then G is either the alternating group or the group symmetric.

Jordan's theorem involving a p-cycle

Theorem (Jordan)

Let G be a primitive permutation group on a finite set Ω. Let p be a prime with $p \leq |\Omega| - 3$. If G contains a p-cycle, then G is either the alternating group or the group symmetric.

Proof Sketch

Let $|\Omega| = n$ and suppose that $p \neq 3$. Let σ be an p-cycle in G.

Define $\Lambda := \{\omega \in \Omega \mid \omega \cdot \sigma = \omega\}$.

We have $|\Omega - \Lambda| = n - |\Lambda| = p$.

Note that $\sigma \in G_\Lambda$, then G_Λ acts transitively on $\Omega - \Lambda$. Since p is a prime number the action is primitive.

Jordan's theorem involving a p-cycle

Proof Sketch

From Jordan's Theorem G is $(|\Lambda| + 1)$-transitive, that implies G is $|\Lambda|$-transitive.

Each element in G induces a permutation in $\mathrm{Sym}(\Lambda)$. Thus the natural homomorphism $\rho : G_{(\Lambda)} \to \mathrm{Sym}(\Lambda)$ is surjective, and then $\mathrm{Sym}(\Lambda) \cong G_{(\Lambda)}/G_{\Lambda}$.

Let $P := \langle \sigma \rangle$. We have $|P| = p$ and $P \subseteq G_{\Lambda}$.

Since G_{Λ} fixes all points in Λ and swaps all points in $\Omega - \Lambda$ we obtain that G_{Λ} acts faithfully on $\Omega - \Lambda$.

Jordan's theorem involving a p-cycle

Proof Sketch

We have $|G_\Lambda|$ divides $p!$. It follows that P is a Sylow p-subgroup of G_Λ.

Observe that $G_\Lambda \trianglelefteq G_{(\Lambda)}$, by the Frattini argument $G_{(\Lambda)} = NG_\Lambda$, where $N = N_{G_{(\Lambda)}}(P)$.

Also observe that $G_{(\Lambda)}/G_\Lambda = NG_\Lambda/G_\Lambda \cong N/(N \cap G_\Lambda)$. That implies $\mathrm{Sym}(\Lambda) \cong N/N_\Lambda$.

Since $|\Lambda| = |\Omega| - p \geq 3$, then $\mathrm{Sym}(\Lambda)$ contains a 3-cycle, and such a 3-cycle must lie in $\mathrm{Alt}(\Lambda)$.

Jordan's theorem involving a p-cycle

Proof Sketch

Note that $(N/N_\Lambda)' = N'N_\Lambda/N_\Lambda \cong N'/(N' \cap N_\Lambda)$.

Thus there exists $x \in N'$ such that the permutation induced by x on Λ is a 3-cycle.

We have $\mathrm{Aut}(P)$ is abelian.

Define $\rho : N \to \mathrm{Aut}(P)$, where $\rho_n(\sigma) = n^{-1}\sigma n$. Since $\ker \rho = C_N(P)$, then $N/C_N(P) \cong \mathrm{Im}\,\rho \leq \mathrm{Aut}(P)$ and $N/C_N(P)$ is an abelian group.

Observe that $x \in N' \subseteq C_N(P)$, and so x commutes with σ.

Jordan's theorem involving a p-cycle

Proof Sketch

The permutation of $\Omega - \Lambda$ induced by x commutes with a p-cycle on these p points. The order of this permutation divides p.

Since x induces a 3-cycle on Λ, and $3 \nmid p$, it follows that x^p also induces a 3-cycle on Λ.

We know that x^p fixes the points of $\Omega - \Lambda$. Since G acts faithfully on Ω, x^p is a 3-cycle.

The corollary give us the result.

Application

Theorem (Bochert)

Let Ω be a finite set of cardinality n and let $S = \mathrm{Sym}(\Omega)$. Let $G \lneq S$ that is primitive on Ω. If $G \neq \mathrm{Alt}(\Omega)$, then $|S : G| \geq (\frac{n+1}{2})!$.

Application

Theorem (Bochert)

Let Ω be a finite set of cardinality n and let $S = \mathrm{Sym}(\Omega)$. Let $G \lneq S$ that is primitive on Ω. If $G \neq \mathrm{Alt}(\Omega)$, then $|S : G| \geq (\frac{n+1}{2})!$.

If G a group of permutation that acts primitively in Ω such that $G \neq \mathrm{Sym}(\Omega), \mathrm{Alt}(\Omega)$, then G is “small”.

Generalizations

Theorem (Neumman, 1975)

Let G be a primitive permutation group of degree n containing a cycle of length p^r, where $p \neq 3$ is a prime number. If $n > p^r + 2$, then G is alternating group or symmetric group.

Generalizations

Theorem (Neumman, 1975)

Let G be a primitive permutation group of degree n containing a cycle of length p^r, where $p \neq 3$ is a prime number. If $n > p^r + 2$, then G is alternating group or symmetric group.

- There is another generalization of the year 2012. This illustrates the importance of the main result of this work.

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