Supersolvable Groups

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Introduction

- A supersolvable group is a group “composed" of cyclic groups, which means that the group has a normal chain with cyclic factors.
- We are going to show some properties of supersolvable groups and their relation with the class of nilpotent groups. As an example, we have the following relations between groups:

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- We are going to show some properties of supersolvable groups and their relation with the class of nilpotent groups. As an example, we have the following relations between groups:

$$\begin{array}{ccccc}
\begin{array}{c} \textit{finitely generated} \\ \textit{nilpotent} \end{array} & \Longrightarrow & \textit{supersolvable} & \Longrightarrow & \textit{solvable} + \textit{maximal condition} \\
\Downarrow & & & & \Downarrow \\
\textit{nilpotent} & & \Longrightarrow & & \textit{solvable}
\end{array}$$

Definition

- A group G is called *supersolvable* if it possesses a finite normal series $G = G_0 \geq G_1 \geq ... \geq G_n \geq G_{n+1} = 1$, in which each factor group G_i/G_{i+1} is cyclic, $\forall \quad 1 \leq i \leq n$.

Example

$G := \langle (1,2,3,4),(1,3) \rangle \cong D_4$ is supersolvable

Consider $G_1 = \langle (1,2,3,4) \rangle$ and $G_2 = \langle (1,3)(2,4) \rangle$, then $G = G_0 \geq G_1 \geq G_2 \geq G_3 = 1$ is a normal chain with cyclic factors, which means that D_4 is supersolvable.

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$S_3 = \langle (1,2,3), (1,2) \rangle$ is supersolvable.

We can check that $S_3 \geq A_3 \geq 1$ is a supersolvable series, then S_3 is supersolvable.

Example

Finitely generated abelian groups are supersolvable.

In fact, let $G = \langle g_1, g_2, \ldots, g_n \rangle$ be an abelian group and define $G_i = \langle g_i, g_{i+1}, \ldots, g_n \rangle$. Since $G_i/G_{i+1} = \langle g_i G_{i+1} \rangle$ and $G_i \trianglelefteq G$, $\forall\ 1 \leq i \leq n$, we have a normal chain $G = G_1 \geq G_2 \geq \ldots \geq G_n = 1$ with cyclic factors. Then G is supersolvable.

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Example

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A_4 is not a supersolvable group.

In fact, K_4 is the only normal subgroup of A_4, then it is not supersolvable.

Remark

- Notice that supersolvable groups are solvable, since G_i/G_{i+1} is abelian;
- Solvability does not imply supersolvability;
  For example, S_4 has a solvable chain $S_4 \trianglerighteq A_4 \trianglerighteq K_4 \trianglerighteq 1$, but it has only these normal subgroups, A_4 and K_4, then it is not supersolvable.

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Proposition

Supersolvable groups are finitely generated.

Proof.

Let $G = G_0 \geq G_1 \geq \ldots \geq G_n \geq G_{n+1} = 1$ be a supersolvable series. For each i, take $g_i \in G_i$ such that $G_i/G_{i+1} = \langle g_i G_{i+1} \rangle$. If $g \in G$, then $g = g_0^{e_0} a_1$ where $a_1 \in G_1$ and $e_0 \in \mathbb{Z}$. Besides that, $a_1 = g_1^{e_1} a_2$ with $a_2 \in G_2$. Proceeding this reasoning, we can write $g = g_0^{e_0} g_1^{e_1} \cdots g_n^{e_n} a_{n+1}$ where $a_{n+1} \in G_{n+1}$, which means $a_{n+1} = 1$. Therefore, $g = g_0^{e_0} g_1^{e_1} \cdots g_n^{e_n}$ then $G = \langle g_0, \ldots, g_n \rangle$. □

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□

Properties

Theorem

Suppose $H \leq G$, $N \trianglelefteq G$, where G is supersolvable group. Then H and G/N are supersolvable.

Proof

First, we shall prove that H is supersolvable.

Consider $G = G_0 \geq G_1 \geq \ldots \geq G_n \geq G_{n+1} = 1$ the supersolvable series of G. Since each $G_i \trianglelefteq G$, defining $H_i = H \cap G_i$ we have $H_i \trianglelefteq H$ and so we get a normal series of H:

$$H = H_0 \geq H_1 \geq \ldots \geq H_n \geq H_{n+1} = 1.$$

Without loss of generality, we may assume that all the factors in the chain above are different.

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Proof

Analyzing its factors:

$$\begin{aligned}(H \cap G_i)/(H \cap G_{i+1}) &= (H \cap G_i)/((H \cap G_i) \cap G_{i+1}) \\ &\cong (H \cap G_i)G_{i+1}/G_{i+1} \\ &\leq G_i/G_{i+1}.\end{aligned}$$

Since G_i/G_{i+1} is cyclic we have, H_i/H_{i+1} is cyclic and H is supersolvable.

Now, we shall prove that G/N is supersolvable. In fact, since $N \trianglelefteq G$, the subgroups G_iN are normal in G and so by the Correspondence Theorem we have a normal series of G/N:

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Properties

Proof.

$$\begin{aligned} G/N = G_0N/N \geq G_1N/N \geq \dots \\ \geq G_nN/N \geq G_{n+1}N/N = N/N. \end{aligned}$$

Using the isomorphism theorems:

$$\begin{aligned} \big(G_iN/N\big)/\big(G_{i+1}N/N\big) &\cong G_iN/G_{i+1}N \\ &= G_iG_{i+1}N/G_{i+1}N \\ &\cong G_i/\big(G_i \cap G_{i+1}N\big) \end{aligned}$$

Since $G_i/\big(G_i \cap G_{i+1}N\big) \leq G_i/G_{i+1}$, then $\big(G_iN/N\big)/\big(G_{i+1}N/N\big)$ is cyclic. Hence G/N is supersolvable.

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Properties

Theorem

The following statements are true:

1. *A direct product of finitely many supersolvable groups is supersolvable.*
2. *If* $H_1, \ldots, H_n$ *are normal subgroups of* G *and the groups* $G/H_1, \ldots, G/H_n$ *are supersolvable, then* $G/\bigcap_{i=1}^{n} H_i$ *is supersolvable.*

Properties

Proof

Let us prove the statement 1. By using induction, it is enough to show that if G and K are supersolvable then so is $G \times K$. Consider a supersolvable series $G = G_0 \geq G_1 \geq \ldots \geq G_n \geq G_{n+1} = 1$ and $K = K_0 \geq K_1 \geq \ldots \geq K_m \geq K_{m+1} = 1$ of G and K, respectively. We have $G_i \times 1 = G_i \times K_{m+1} \trianglelefteq G \times K$ and $G \times K_j = G_0 \times K_j \trianglelefteq G \times K$. Furthemore,

$$(G_i \times 1)/(G_{i+1} \times 1) \cong G_i/G_{i+1} \times 1/1 \cong G_i/G_{i+1}$$

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is a supersolvable series of $G \times K$.

For the statement 2, consider the homomorphism

$$\phi : G \longrightarrow \prod_{i=1}^{n} G/H_i, \; g \mapsto (gH_1, \ldots, gH_n)$$

whose kernel is $\bigcap_{i=1}^{n} H_i$. It follows that $G/\bigcap_{i=1}^{n} H_i \cong \prod_{i=1}^{n} G/H_i$ which is a supersolvable group by the statement 1. □

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Corollary

Supersolvable groups satisfy the maximal condition of chains of subgroups.

Proof

Let G be a supersolvable group and suppose that it has a infinite series of subgroups

$$H_0 \leq H_1 \leq \ldots \leq H_n \leq \ldots$$

Since subgroups of supersolvable groups are supersolvable then all subgroups of G are finitely generated. In particular $\bigcup_{i=0}^{\infty} H_i \leq G$ is finitely generated.

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Remark

Proof.

Assume it is generated by $\{x_1, \ldots, x_k\}$ then there exists $m \in \mathbb{N}$ such that $\{x_1, \ldots, x_k\} \subseteq H_m$ which means $\bigcup_{i=0}^{\infty} H_i \subseteq H_m$, a contradiction. □

- We know that all nilpotent groups are solvable;
- Supersolvability (then solvable) does not imply nilpotence (Consider S_3);
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The next result gives us a relation between nilpotent and supersolvable groups:

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Proof.

Assume it is generated by $\{x_1, \ldots, x_k\}$ then there exists $m \in \mathbb{N}$ such that $\{x_1, \ldots, x_k\} \subseteq H_m$ which means $\bigcup_{i=0}^{\infty} H_i \subseteq H_m$, a contradiction. □

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The next result gives us a relation between nilpotent and supersolvable groups:

Properties

Proposition

If G is a finitely generated nilpotent group, then G is supersolvable.

Proof

Consider a upper central series

$1 = \zeta_0 \leq \zeta_1 \leq \cdots \leq \zeta_c \leq \zeta_{c+1} = G$, where $\zeta_i \trianglelefteq G$ and $\frac{\zeta_{i+1}}{\zeta_i} \leq Z\left(\frac{G}{\zeta_i}\right)$. Notice that ζ_{i+1}/ζ_i is abelian and finitely generated, thus we have it is a direct product of cyclic groups

$$\frac{\zeta_{i+1}}{\zeta_i} = \frac{C_{i_1}}{\zeta_i} \times \frac{C_{i_2}}{\zeta_i} \times \cdots \times \frac{C_{i_k}}{\zeta_i}.$$

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The previous proposition show us directly that all finite p-groups and finitely generated abelian groups are supersolvable.

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The previous proposition show us directly that all finite p-groups and finitely generated abelian groups are supersolvable.

Properties

It is not true that $N \trianglelefteq G$ and G/N supersolvable implies G supersolvable. For example, A_4 is not supersolvable, but K_4 and $A_4/K_4 \cong C_3$ are supersolvable.

Definition

Let G be a group and $N \trianglelefteq G$. We say that N is $G-$supersolvable if N has a supersolvable series $N = N_0 \geq N_1 \geq \ldots \geq N_n \geq N_{n+1} = 1$ with $N_i \trianglelefteq G$, for all $0 \leq i \leq n+1$.

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Let $N \trianglelefteq G$. If N is $G-$supersolvable and G/N is supersolvable then G is supersolvable.

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In fact, applying the Correspondence Theorem to a supersolvable series of G/N we can write

$$G/N = G_0/N \geq G_1/N \geq \ldots \geq G_n/N \geq G_{n+1}/N = N/N,$$

where each G_i is normal in G with $N \leq G_i$. We get that

$$G = G_0 \geq G_1 \geq \ldots \geq G_n \geq N$$

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Since N is $G-$supersolvable we have

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Properties

Proposition

If G is a supersolvable group and $N \trianglelefteq G$, then N occurs as a term in a supersolvable series of G.

Proof

As the quotient G/N is supersolvable, using the Correspondence Theorem we can get a normal series of G from G up to N:

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where G_{i+1}/G_i is cyclic.

Take $G = H_0 \geq H_1 \geq \ldots \geq H_r \geq H_{r+1} = 1$ a supersolvable series of G and define $N_i = H_i \cap N$.

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Since a supersolvable group is closed to subgroups we get $N = N_0 \geq N_1 \geq \ldots \geq N_r \geq N_{r+1} = 1$ supersolvable series of N. Then N occurs in the following supersolvable series of G:

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In particular, the previous propositions says that if N is a normal cyclic subgroup of G with G/N supersolvable, then G is supersolvable.

An immediate consequence of proposition is that normal subgroups of a supersolvable group G are G−supersolvable.

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Properties

Lemma

1. *Every subgroup of an infinite cyclic group different from the identity is an infinite cyclic group of finite index, and there is a unique subgroup for each finite index.*
2. *Every subgroup of a finite cyclic group of order n is a cyclic group of order dividing n, and there is a unique subgroup of each order dividing n.*

Properties

Theorem

A supersolvable group G has a normal series $G = G_0 \geq G_1 \geq \ldots \geq G_n \geq G_{n+1} = 1$ in which every factor group G_i/G_{i+1} is either infinite cyclic or cyclic of prime order.

Proof

Let G be supersolvable with supersolvable series $G = G_0 \geq G_1 \geq \ldots \geq G_n \geq G_{n+1} = 1$. If G_i/G_{i+1} is cyclic of finite order $p_1 p_2 \cdots p_s$ where $p_1, \ldots, p_s$ are primes (not necessarily distinct) then, by the previous lemma, G_i/G_{i+1} has only one subgroup $H_{i\,1}/G_{i+1}, \ldots, H_{i\,s}/G_{i+1}$ of order $p_1 p_2 \cdots p_s$, $p_2 \cdots p_s, \ldots, p_s$ respectively.

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Since $H_{i\,j}/G_{i+1}$ is a characteristic subgroup of $G_i/G_{i+1} \trianglelefteq G/G_{i+1}$ we have $H_{i\,j}/G_{i+1} \trianglelefteq G/G_{i+1}$ then $H_{i\,j} \trianglelefteq G$. Hence, we can refine our series as below

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Notice, by the Isomorphism Theorem that $H_{i\,j}/H_{i,j+1} \cong \dfrac{H_{i\,j}/G_{i+1}}{H_{i,j+1}/G_{i+1}}$. Then $|H_{i\,j}/H_{i\,j+1}| = p_j$. If we repeat this process for all $0 \leq i \leq n+1$ we get a normal series of G with infinite cyclic factors or cyclic of prime order. □

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Properties

Moreover, we can refine our supersolvable series with infinite cyclic factors or cyclic of prime order, and if G_{i-1}/G_i and G_i/G_{i+1} are of prime orders p_i and p_{i+1} we have $p_i \leq p_{i+1}$.

Corollary

A supersolvable group has a cyclic normal subgroup of infinite or prime order.

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A simple supersolvable group is cyclic of prime order.

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Properties

Definition

A chief factor of G is a quotient H/K where $H, K \trianglelefteq G$ and H/K is a minimal normal subgroup of G/K. A chief series of G is a normal series whose factors are chief.

Properties

Proposition

The following statements are true:

1. *A minimal normal subgroup of a supersolvable group is cyclic of prime order.*
2. *A chief factor of a supersolvable group is cyclic of prime order.*
3. *A supersolvable group with a chief series is a finite group.*
4. *G is a finite supersolvable group if and only if it has a chief series with cyclic factors of prime order.*

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Properties

Proof

1. Let N be a minimal normal subgroup of G. N is $G-$supersolvable. By minimality, N is simple. Applying a previous Corollary, we have that N is cyclic of prime order.
2. If H/K is a chief factor of G, then H/K is a minimal normal subgroup of G/K. Quotient groups of G are supersolvable and thus by 1, H/K has prime order.

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4. Finally, note that a finite group has chief series and thus a finite supersolvable group has a chief series with cyclic of prime order factors by 1. Conversely, a chief series with cyclic factors is a normal series with cyclic factors and hence is a supersolvable series.

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Properties

Theorem

Let G be a supersolvable group. Then all maximal subgroups of G has prime index.

Proof

Consider $H \underset{max}{<} G$. If $H \trianglelefteq G$, G/H is a simple and supersolvable group, therefore G/H is a cyclic group with prime order, then $[G : H]$ is prime.

Now, suppose that H is not a normal subgroup of G. Let K be the maximal subgroup of H that is normal in G; we have that $H/K \underset{max}{<} G/K$ and $[G : H] = [\frac{G}{K} : \frac{H}{K}]$, thus we may assume that $K = 1$, without loss of generality.

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Consider $H \underset{max}{<} G$. If $H \trianglelefteq G$, G/H is a simple and supersolvable group, therefore G/H is a cyclic group with prime order, then $[G:H]$ is prime.

Now, suppose that H is not a normal subgroup of G. Let K be the maximal subgroup of H that is normal in G; we have that $H/K \underset{max}{<} G/K$ and $[G:H] = \left[\frac{G}{K} : \frac{H}{K}\right]$, thus we may assume that $K = 1$, without loss of generality.

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Proof

Since G is supersolvable, we conclude that G has a normal subgroup A, which is infinite cyclic or cyclic of prime order.

If A is infinite cyclic then $A \cong \mathbb{Z}$; since $Aut(\mathbb{Z}) \cong \mathbb{Z}_2$, all subgroups of $\mathbb{Z}$ are characteristic.

Since $A \cap H \leq A$, then $A \cap H \trianglelefteq G$ therefore $A \cap H = 1$ because of $K = 1$ is the biggest subgroup with this property. Moreover, we can choose B a proper subgroup of A that is normal in G. Then we have $H < AH$ and, by the maximality of H, $AH = G$. However, this means that $H < BH < AH = G$, which contradicts the maximality of H.

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Properties

Proof.

Thus A has prime order. Then we have

$$[G : H] = [AH : H] = [A : A \cap H] = [A : 1] = |A|$$

This means that H has prime index. □

Properties

The next result assures us the reciprocal of the previous theorem in the finite case.

Theorem (Huppert)

Suppose G is a finite group with the property that all its maximal subgroups are of prime index. Then G is supersolvable.

Aplication

Theorem

Let G be a supersolvable group; then the commutator subgroup G' of G is nilpotent.

Proof

Let $G = G_0 \geq G_1 \geq \ldots \geq G_n \geq G_{n+1} = 1$ be a supersolvable series. Defining $H_i := G' \cap G_i$, as we have already observed, $G' = H_0 \geq H_1 \geq \ldots \geq H_{n+1} = 1$ is a supersolvable series for G' and moreover, $H_i \trianglelefteq G$. Let K_j be the distinct terms of this chain, we affirm that $G' = K_0 \geq K_1 \geq \ldots \geq K_s = 1$ is a central series for G'. In fact, we already checked that the $K's$ form a normal series, now we will verify that $K_i/K_{i+1} \leq Z(G'/K_{i+1})$. This is equivalent to show that $[K_i, G'] \leq K_{i+1}$.

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Proof.

Note that $g \in G$ induces an automorphism $\varphi_g \in Aut(K_i/K_{i+1})$ in the cyclic group K_i/K_{i+1} defined by $(a_iK_{i+1})\varphi_g = a_i^g K_{i+1}$. Since the group of automorphism of a cyclic group is abelian, the automorphism induced by the element $[x, y]$ is trivial, with $x, y \in G$, then we have

$$(a_iK_{i+1})\varphi_{[x,y]} = a_i^{[x,y]}K_{i+1} = a_iK_{i+1}$$

Therefore $a_i^{-1}a_i^{[x,y]} = [a_i, [x, y]] \in K_{i+1}$, $\forall\, a_i \in K_{i+1}$ and $x, y \in G$, then $[K_i, G'] \leq K_{i+1}$, as we wish. □

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Aplications

- We have already checked that S_3 is supersolvable but is not nilpotent, however $S_3' = A_3$ is nilpotent.
- The converse of the last result is not true, since $A_4' = K_4$ is nilpotent and A_4 is not supersolvable.

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Thank you!