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Fitting subgroup

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Definition

Let G be group and p be a prime.

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Proposition

1. $O_p(G) = \bigcap_{P \in Syl_p(G)} P$;
2. $O_p(G)\ char\ G$;
3. $O_p(G)$ *is the largest normal p-subgroup of G;*

Definition

The Fitting *subgroup of a group G is defined as*

$$F(G) = \langle N \trianglelefteq G \mid N \textit{ is nilpotent} \rangle.$$

- There are in the literature another definitions for the Fitting subgroup.
- If G is nilpotent, then $F(G) = G$. In particular, if G is abelian, $F(G) = G$.

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3. $F(G) = \bigtimes_{p \in \pi(G)} O_p(G)$ *(internal direct product)*;
4. $F(G)$ *is the largest nilpotent normal subgroup of* G.

Corollary - Fitting's Theorem

If H and K are nilpotent normal subgroups of a finite group G, then also HK is a normal nilpotent subgroup of G.

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Corollary

The subgroup $F(G)$ of a group G may be described as the unique maximal nilpotent normal subgroup of G.

Remark

If G is a solvable group, then its Fitting subgroup is nontrivial.

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Fitting subgroup of S_n

- **For $n = 2$:** $S_2 \simeq \mathbb{Z}_2$ is nilpotent. Then $F(S_2) = S_2$.

Remark: For $n \geq 3$ we have $Z(S_n) = 1$, then S_n is non-nilpotent and $F(S_n) < S_n$.

- **For $n = 3$:**
  - Normal subgroups of S_3: $1, A_3$ and S_3;
  - $A_3 \simeq \mathbb{Z}_3$ is nilpotent.

  Then $F(S_3) = A_3$.

Remark: For $n \geq 4$ we have $Z(A_n) = 1$, then A_n is non-nilpotent and $F(A_n) < A_n$.

- **For $n = 4$:**
  - Normal subgroups of S_4: $1, A_4, X = \langle (12)(34), (13)(24) \rangle$ and S_4;
  - $X \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ is nilpotent.

  Then $F(S_4) = X$.

- **For $n \geq 5$:**
  - Normal subgroups of S_n: $1, A_n$ and S_n.

  Then $F(S_n) = 1$.

Fitting subgroup of D_n

Remark: D_n is nilpotent if and only if $n = 2^j$ for some $j \geq 0$.

- $F(D_n) = D_n$ if and only if $n = 2^j$ for some $j \geq 0$.

- Suppose $n = 2^j m$ for some $j \geq 0$ and $m \geq 3$ odd.

$$D_n = \left\langle a, b \mid a^n = b^2 = 1\,, bab = a^{-1} \right\rangle$$

  - Normal subgroups of D_n: $1, \left\langle a^d \right\rangle$ for some $d \mid n$. In addiction, if n is even, $G = \left\langle a^2, ab \right\rangle$ and $H = \left\langle a^2, b \right\rangle$ are also normal. These are all proper and normal subgroups of D_n.

- $\langle a \rangle \trianglelefteq D_n$ is nilpotent, then $\langle a \rangle \leq F(D_n)$.
    - $$\langle a \rangle \leq F(D_n) = \begin{cases} 1 \\ \langle a \rangle \\ G = \left\langle a^2, ab \right\rangle, n \text{ even} \\ H = \left\langle a^2, b \right\rangle, n \text{ even} \end{cases}$$
- Then $F(D_n) = \langle a \rangle$ if n is not a power of two.

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$$1.F(G)\, char\, G$$

Proof:

- Suppose $\alpha \in Aut(G)$
- $\alpha(F(G)) = \prod_{p\in\pi(G)} \alpha(O_p(G))$
- For each $p \in \pi(G)$, $O_p(G)\, char\, G$
- Then $\prod_{p\in\pi(G)} O_p(G) = F(G)$

∎

2. If $N \trianglelefteq G$, then $N \cap F(G) = F(N)$

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Proof:

- $F(N)\, char\, N \trianglelefteq G$, then $F(N) \trianglelefteq G$
- Note that $F(N) \leqslant F(G)$. Thus $F(N) \leq N \cap F(G)$
- $N \cap F(G) \trianglelefteq G$ and is nilpotent, because $N \cap F(G) \leqslant F(G)$.
- Thus $N \cap F(G) \leq F(N)$.
- Hence $N \cap F(G) = F(N)$

■

Preliminar result

Theorem

Let G be a group, $Z \leq Z(G)$. Then G/Z is nilpotent if, and only if, G is nilpotent.

Corollary

Let G be a group, $Z \leq Z(G)$ and $H \leq G$ such that $Z \leq H$. Then H/Z is nilpotent if, and only if, H is nilpotent.

3. If $Z \leq Z(G)$, then $F(G/Z) = F(G)/Z$.

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Proof:

- Write $F(G/Z) = H/Z$, where $Z \leq H \leq G$.
- By the Correspondece Theorem $H \trianglelefteq G$ and $Z \leqslant H$.
- So, by the previous Corollary, H is nilpotent.
- So $H \leq F(G)$ and this implies $F(G/Z) = H/Z \leq F(G)/Z$
- $Z \leq F(G)$, by previous Corollary, $F(G)/Z$ is nilpotent. Note that $F(G) \trianglelefteq G$.
- Hence $F(G)/Z \trianglelefteq G/Z$. Thus $F(G)/Z \leq F(G/Z)$.

Fitting subgroup of $PSL(n, \mathbb{F})$

Let $(n, |\mathbb{F}|) \neq (2,2), (2,3)$ and $\mathbb{F}$ is a finite field.

$$PSL_n(\mathbb{F}) = SL_n(\mathbb{F})/Z(SL_n(\mathbb{F}))$$

- Note that $SL_n(\mathbb{F})$ is perfect, so is non-nilpotent.

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$$PSL_n(\mathbb{F}) \text{ is simple } \implies \begin{cases} F(PSL_n(\mathbb{F})) = 1 \\ F(PSL_n(\mathbb{F})) = PSL_n(\mathbb{F}) \end{cases}$$

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which implies $SL_n(\mathbb{F})$nilpotent.

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which implies $SL_n(\mathbb{F})$nilpotent.
Then $F(PSL_n(\mathbb{F})) = 1$ and $F(SL_n(\mathbb{F})) = Z(SL_n(\mathbb{F}))$.

Properties of Fitting subgroup

Theorem

Let G be a group and $\Phi(G)$ its Frattini subgroup. Then:

1. $[F(G), F(G)] \leq \Phi(G) \leq F(G)$.
2. $F(G/\Phi(G)) = F(G)/\Phi(G)$.

Contents

Definition

A **Fitting series**, or nilpotent series, of a group G is a normal series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_{n-1} \leq G_n = G$$

such that each factor G_{i+1}/G_i is nilpotent.

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Upper Fitting series

For a group G, define:

- $F_0 = 1$
- $F_1 = F(G)$
- F_{i+1} is such that $F_{i+1}/F_i = F(G/F_i)$

$$1 = F_0 \leq F_1 \leq \cdots \leq F_{n-1} \leq F_n = G$$

Fitting Series of $G = S_4$

- $F_0 = 1, F_1 = F(S_4) = X := \langle (12)(34), (13)(24) \rangle$.
- F_2 is such that $F_2/X = F(S_4/X)$.
  - Note that S_4/X is non-abelian with order 6, then $S_4/X \simeq S_3$.
- Then
$$F_2/F_1 = F_2/X = F(S_4/X) \simeq F(S_3) = A_3.$$
  The only subgroup of S_4 with order 12 is A_4. Thus $F_2 = A_4$.
- $F_3/F_2 = F_3/A_4 = F(S_4/A_4) = S_4/A_4$. Therefore $F_3 = S_4$.
$$1 < X < A_4 < S_4$$

Fitting series of $G = D_n$

$$D_n = \langle a, b : a^n = b^2 = 1\,, bab = a^{-1} \rangle$$

- If $n = 2^j$ for some $j \in \mathbb{N}$, then $F(D_n) = D_n$.

$$1 < D_n$$

- Suppose $n = 2^j m$ for some $j \in \mathbb{N}$ and $m \geq 3$ odd.
  $F_1 = F(D_n) = \langle a \rangle$.
  $F_2/\langle a \rangle = F(D_n/\langle a \rangle) = D_n/\langle a \rangle$. Then $F_2 = D_n$.

$$1 < \langle a \rangle < D_n$$

Fitting Length

Theorem

Suppose that

$$1 = G_0 \leq G_1 \leq \cdots \leq \ldots G_{n-1} \leq G_n = G$$

is a nilpotent series, then $G_i \leq F_i \; \forall i$.

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Definition

We define the **Fitting length** as the smallest j for which $F_j = G$. If G is trivial, G has Fitting length 0. If $F_i \neq G$ for all i, G has no Fitting length.

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Definition

We define the **Fitting length** as the smallest j for which $F_j = G$. If G is trivial, G has Fitting length 0. If $F_i \neq G$ for all i, G has no Fitting length.

Theorem

A finite group G is solvable if, and only if, has Fitting length.

Proof

Suppose G is solvable.

- Since G is finite, $F_k = F_{k+1}$ for some k.

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Suppose G is solvable.

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- $F(G/F_k) = F_{k+1}/F_k = 1$

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- G is solvable, then G/F_k also.

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- A non-trivial solvable group has non-trivial Fitting subgroup.

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- Since G is finite, $F_k = F_{k+1}$ for some k.
- $F(G/F_k) = F_{k+1}/F_k = 1$
- G is solvable, then G/F_k also.
- A non-trivial solvable group has non-trivial Fitting subgroup.
- Then $G = F_k$ and G has Fitting length.

Proof

Suppose that G has Fitting length j.
We claim that F_i is solvable for each i. By induction:

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- For $i = k$, suppose F_k is solvable.
- F_{k+1}/F_k is solvable. Then F_{k+1} is also. So the claim is proved.
- There exists j such that $F_j = G$. Hence G is solvable.

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Moral

- Criterion of solubility.
- Not every solvable group is nilpotent. However, in finite case, Fitting length intuitively measures how far such group is to be nilpotent.
- In general, to study solvable groups by nilpotent factors of its Fitting series.

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